

Lower Bounds on the Probability of a Finite Union of Events[☆]

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Abstract

In this paper, lower bounds on the probability of a finite union of events are considered, *i.e.* $P\left(\bigcup_{i=1}^N A_i\right)$, in terms of the individual event probabilities $\{P(A_i), i = 1, \dots, N\}$ and the *sums* of the pairwise event probabilities, *i.e.*, $\{\sum_{j:j \neq i} P(A_i \cap A_j), i = 1, \dots, N\}$. The contribution of this paper includes the following: (i) in the class of all lower bounds that are established in terms of only the $P(A_i)$'s and $\sum_{j:j \neq i} P(A_i \cap A_j)$'s, the *optimal* lower bound is given numerically by solving a linear programming (LP) problem with $N^2 - N + 1$ variables; (ii) a new analytical lower bound is proposed based on a relaxed LP problem, which is at least as good as the bound due to Kuai, et al. [1]; (iii) numerical examples are provided to illustrate the performance of the bounds.

Keywords: Finite Union of Events, Lower and Upper Bounds, Optimal Bounds, Linear Programming.

1. Introduction

Lower and upper bounds of $P\left(\bigcup_{i=1}^N A_i\right)$ in terms of the individual event probabilities $P(A_i)$'s and the pairwise event probabilities $P(A_i \cap A_j)$'s can be seen as special cases of the Boolean probability bounding problem [2, 3], which can be solved numerically via a linear programming (LP) problem involving 2^N variables. Unfortunately, the number of variables for Boolean

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probability bounding problems increases exponentially with the number of events, N , which makes finding the solution impractical. Therefore, some suboptimal numerical bounds are proposed [2, 3, 4, 5] in order to reduce the complexity of the LP problem, for example, by using the dual basic feasible solutions.

On the other hand, analytical lower bounds are particularly important. The Kuai-Alajaji-Takahara (KAT) bound [1] is one of the analytical lower bounds that has been shown to be better than the Dawson-Sankoff (DS) bound [6] and D. de Caen's bound [7]. These analytical bounds are later investigated in other works (e.g., see [8, 9, 10, 11, 12, 13]).

As in [7], the KAT lower bound [1] for $P\left(\bigcup_{i=1}^N A_i\right)$ is expressed in terms of only $\sum_{j:j \neq i} P(A_i \cap A_j)$'s and $P(A_i)$'s, and hence knowledge of the individual pairwise event probabilities $P(A_i \cap A_j)$ is not required. In this paper, we revisit and investigate the same problem that lower bounds are established in terms of only the *sums* of the pairwise event probabilities, *i.e.*, $\sum_{j:j \neq i} P(A_i \cap A_j)$, and the individual event probabilities $P(A_i)$'s, without the use of the $P(A_i \cap A_j)$'s.

Our contributions are the following. First, in the class of all lower bounds that are expressed in terms of only the $P(A_i)$'s and the $\sum_{j:j \neq i} P(A_i \cap A_j)$'s, the *optimal* lower bound is proposed numerically by solving an LP problem, which has only $N^2 - N + 1$ variables. Here *optimality* means that any lower bound for $P\left(\bigcup_{i=1}^N A_i\right)$ in terms of only $\sum_{j:j \neq i} P(A_i \cap A_j)$'s and $P(A_i)$'s cannot be sharper than the proposed optimal lower bound. This is proven by showing that the proposed optimal lower bound can always be achieved by constructing $\{A_i, i = 1, \dots, N\}$ that satisfy all known information on the $\sum_{j:j \neq i} P(A_i \cap A_j)$'s and $P(A_i)$'s. The computational complexity of the optimal lower bound is low since the number of variables are not exponentially increasing in N . Next, a suboptimal analytical lower bound is established by solving a relaxed LP problem. The new analytical bound is proven to be at least as good as the existing KAT bound [1]. Finally, we analyze the performance of the new bounds by comparing them with the KAT bound and other existing bounds. In particular, numerical results show that the Gallot-Kounias (GK) bound [14, 15], which was recently revisited in [16, 17], is not necessarily sharper than the proposed lower bounds as well as the KAT bound (see also [18] for another example), even though it exploits full information of all $P(A_i \cap A_j)$'s and $P(A_i)$'s.

2. Main Results

Consider a finite family of events A_1, \dots, A_N in a general probability space (Ω, \mathcal{F}, P) , where N is a fixed positive integer. Note that there are only finitely many Boolean atoms¹ specified by the A_i 's [7]. For each atom $\omega \in \mathcal{F}$, let $p(\omega) := P(\omega)$, and let the degree of ω , denoted by $\deg(\omega)$, be the number of A_i 's that contain ω . Define

$$a_i(k) := P(\{\omega \subseteq A_i : \deg(\omega) = k\}), \quad (1)$$

where $i = 1, \dots, N$ and $k = 1, \dots, N$. Then from [1, Lemma 1], we know that

$$P\left(\bigcup_{i=1}^N A_i\right) = \sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k}. \quad (2)$$

In this paper, using the same notation as in [1], lower bounds on $P\left(\bigcup_{i=1}^N A_i\right)$ are established only in terms of $\alpha_i := P(A_i)$ and $\beta_i := \sum_{j:j \neq i} P(A_i \cap A_j)$, $i = 1, \dots, N$. For simplicity, we denote $\gamma_i := \alpha_i + \beta_i$. Then it is easy to verify that the following equalities hold:

$$P(A_i) = \sum_{k=1}^N a_i(k) = \alpha_i, \quad \sum_j P(A_i \cap A_j) = \sum_{k=1}^N k a_i(k) = \gamma_i, \quad i = 1, \dots, N. \quad (3)$$

Let \mathcal{L} denote the set of all lower bounds that are established in terms of only $\{\alpha_i, i = 1, \dots, N\}$ and $\{\gamma_i, i = 1, \dots, N\}$. Then any lower bound in \mathcal{L} , say $\ell \in \mathcal{L}$, is a function of only $\{\alpha_i\}$'s and $\{\gamma_i\}$'s. Also, claiming that $\ell \in \mathcal{L}$ is a lower bound on $P\left(\bigcup_{i=1}^N A_i\right)$ means that for any events $\{A_i, i = 1, \dots, N\}$ that satisfy $P(A_i) = \alpha_i, i = 1, \dots, N$ and $\sum_j P(A_i \cap A_j) = \gamma_i, i = 1, \dots, N$, we must have $P\left(\bigcup_{i=1}^N A_i\right) \geq \ell$.

We first define an *optimal* lower bound in a general class. Assume that each collection $\{A_1, \dots, A_N\}$ of N sets, $A_i \in \mathcal{F}$, is represented by a vector

¹The problem can be directly reduced to the finite probability space case. Thus, through the numerical examples in this paper, we will consider finite probability spaces where $\omega \in \Omega$ denotes an elementary outcome instead of an atom.

$\theta = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m$, which represents partial probabilistic information² about the union $\bigcup_{i=1}^N A_i$. Let Θ denote the set of all possible θ and \mathcal{L}_Θ the set of all lower bounds on $P\left(\bigcup_{i=1}^N A_i\right)$ that are functions of only θ .

Definition 1. We say that a lower bound $\ell \in \mathcal{L}_\Theta$ is achievable if for every $\theta \in \Theta$,

$$\inf_{A_1, \dots, A_N} P\left(\bigcup_{i=1}^N A_i\right) = \ell(\theta),$$

where the infimum ranges over all collections $\{A_1, \dots, A_N\}$, $A_i \in \mathcal{F}$, such that $\{A_1, \dots, A_N\}$ is represented by θ .

Definition 2. We say that a lower bound $\ell^* \in \mathcal{L}_\Theta$ is optimal in \mathcal{L}_Θ if $\ell^*(\theta) \geq \ell(\theta)$ for all $\theta \in \Theta$ and $\ell \in \mathcal{L}_\Theta$.

For bounds in \mathcal{L}_Θ , the following lemma shows that achievability is equivalent to optimality.

Lemma 1. A lower bound $\ell^* \in \mathcal{L}_\Theta$ is optimal in \mathcal{L}_Θ if and only if it is achievable.

Proof. Suppose that ℓ^* is achievable. Let $\theta \in \Theta$ and $\epsilon > 0$ be given, and let ℓ be any lower bound in \mathcal{L}_Θ . By achievability there exist sets A_1, \dots, A_N in \mathcal{F} represented by θ such that

$$\ell^*(\theta) > P\left(\bigcup_{i=1}^N A_i\right) - \epsilon \geq \ell(\theta) - \epsilon.$$

Since this holds for any ϵ we have $\ell^*(\theta) \geq \ell(\theta)$. We prove the converse by the contrapositive. Suppose that ℓ^* is not achievable. Then there exists $\theta' \in \Theta$ such that

$$\inf_{A_1, \dots, A_N} P\left(\bigcup_{i=1}^N A_i\right) > \ell^*(\theta'),$$

²The partial information represented by θ , in which we are interested in this paper, is any m -dimensional linear function of degree- K probabilities of A_1, \dots, A_N , $K = 1, \dots, N$. Here a degree- K probability is defined as $P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_K})$ where i_1, \dots, i_K are K distinct integers in $\{1, 2, \dots, N\}$.

where the infimum ranges over all collections $\{A_1, \dots, A_N\}$, $A_i \in \mathcal{F}$, such that $\{A_1, \dots, A_N\}$ is represented by θ' . Define ℓ by

$$\ell(\theta) = \begin{cases} c & \text{if } \theta = \theta' \\ 0 & \text{if } \theta \neq \theta', \end{cases}$$

where c satisfies

$$\inf_{A_1, \dots, A_N} P \left(\bigcup_{i=1}^N A_i \right) > c > \ell^*(\theta').$$

Then $\ell \in \mathcal{L}_\Theta$ and is larger than ℓ^* at θ' . Hence, ℓ^* is not optimal. ■

Clearly, in our problem, we have $\theta = (\alpha_1, \dots, \alpha_N, \gamma_1, \dots, \gamma_N)$ and $\mathcal{L}_\Theta = \mathcal{L}$. We herein state the following lemma regarding the existing KAT bound.

Lemma 2 (KAT Bound[1]). *The solution of the following LP problem*

$$\begin{aligned} \min_{\{a_i(k), i=1, \dots, N, k=1, \dots, N\}} & \sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \\ \text{s.t.} & \sum_{k=1}^N a_i(k) = \alpha_i, \quad \sum_{k=1}^N k a_i(k) = \gamma_i, \quad i = 1, \dots, N, \\ & a_i(k) \geq 0, \quad i = 1, \dots, N, \quad k = 1, \dots, N, \end{aligned} \quad (4)$$

gives the KAT bound:

$$P \left(\bigcup_{i=1}^N A_i \right) \geq \sum_{i=1}^N \left\{ \left[\frac{1}{\lfloor \frac{\gamma_i}{\alpha_i} \rfloor} - \frac{\frac{\gamma_i}{\alpha_i} - \lfloor \frac{\gamma_i}{\alpha_i} \rfloor}{(1 + \lfloor \frac{\gamma_i}{\alpha_i} \rfloor)(\lfloor \frac{\gamma_i}{\alpha_i} \rfloor)} \right] \alpha_i \right\}, \quad (5)$$

where $\lfloor x \rfloor$ is the largest positive integer less than or equal to x .

Proof. See [1]. ■

Denoting ℓ_{KAT} as the KAT bound in (5), we can see that the KAT bound is a lower bound which is established in terms of only $\{\alpha_i\}$'s and $\{\gamma_i\}$'s. Thus, $\ell_{\text{KAT}} \in \mathcal{L}$. One should note that for a given family of events $\{A_i, i = 1, \dots, N\}$, the $a_i(k)$'s can be obtained from their definition in (1). However, this does not mean that for each feasible point $\{a_i(k)\}$ of the LP problem (4), there exists a corresponding family of events $\{A_i, i = 1, \dots, N\}$. In particular, for the solution of (4), it is possible that a family of events $\{A_i, i = 1, \dots, N\}$ can never be constructed.

Example 1. *Considering a finite probability space (where atoms ω are reduced to elementary outcomes), shown as System V in Table 1, we have*

$$N = 3, \alpha_1 = 0.1, \alpha_2 = \alpha_3 = 0.2, \gamma_1 = 0.21, \gamma_2 = \gamma_3 = 0.265.$$

The KAT solution $\{a_i(k)\}$ for $\ell_{KAT} = 0.3833$ is obtained only at the following optimal feasible point of (4):

$$\begin{aligned} a_1(1) &= 0, a_1(2) = 0.09, a_1(3) = 0.01, a_2(1) = a_3(1) = 0.135, \\ a_2(2) &= a_3(2) = 0.065, a_2(3) = a_3(3) = 0. \end{aligned}$$

However, $a_1(3) := P(\{\omega \in A_1 : \deg(\omega) = 3\}) = 0.01$ implies $P(A_1 \cap A_2 \cap A_3) \geq 0.01$, since $\deg(\omega) = 3$ means that the corresponding outcome ω must be contained in all $A_i, i = 1, \dots, 3$. However, $a_2(3) := P(\{\omega \in A_2 : \deg(\omega) = 3\}) = 0$ implies such ω is not in A_2 , which is a contradiction. Therefore, there is no family of events $\{A_1, A_2, A_3\}$ that can be constructed for this system such that $P(A_1 \cup A_2 \cup A_3) = 0.3833$. In other words, for any sets $\{A_1, A_2, A_3\}$ with given value of $\{\alpha_i\}$'s and $\{\gamma_i\}$'s, we must have $P(A_1 \cup A_2 \cup A_3) > 0.3833$.

Remark 1. *It can be shown that the LP problem (4) has a unique optimal feasible point (see Appendix A). Therefore, the KAT bound is achievable if and only if the optimal feasible point of the LP problem (4) has a corresponding family of events $\{A_i, i = 1, \dots, N\}$ that satisfies the information represented by $\theta = (\alpha_1, \dots, \alpha_N, \gamma_1, \dots, \gamma_N)$. From Example 1, we see that ℓ_{KAT} is not optimal in \mathcal{L} .*

2.1. Optimal Numerical Lower Bound

In order to get a better lower bound than the KAT bound, we herein introduce more constraints on the $a_i(k)$'s in (4) so that the feasible set of $a_i(k)$'s becomes smaller, thus resulting in a sharper lower bound. By Lemma 1, if a family of events $\{A_i\}$ can always be constructed for any feasible point of the resulting LP problem, then the solution must be the optimal lower bound. We establish the numerically computable optimal lower bound in the following theorem.

Theorem 1 (Optimal Numerical Lower Bound). *The optimal lower bound is given by solving the following LP problem:*

$$\begin{aligned}
& \min_{\{a_i(k), i=1, \dots, N, k=1, \dots, N\}} && \sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \\
& s. t. && \sum_{k=1}^N a_i(k) = \alpha_i, \quad \sum_{k=1}^N k a_i(k) = \gamma_i, \quad i = 1, \dots, N, \\
& && \sum_{i=1}^N a_i(k) \geq k a_j(k), \quad j = 1, \dots, N, \quad k = 1, \dots, N, \\
& && a_i(k) \geq 0, \quad i = 1, \dots, N, \quad k = 1, \dots, N.
\end{aligned} \tag{6}$$

Proof. Denote the optimal lower bound in \mathcal{L} by ℓ_{OPT} then $\ell_{\text{OPT}} \in \mathcal{L}$ satisfies $\ell_{\text{OPT}} \geq \ell$ for all $\ell \in \mathcal{L}$. Let the solution of (6) be ℓ'_{OPT} , we will show that $\ell'_{\text{OPT}} = \ell_{\text{OPT}}$. First, it is easy to prove that for any $\{a_i(k)\}$ obtained by (1) from a family of events $\{A_i\}$, the additional constraints $\sum_{i=1}^N a_i(k) \geq k a_j(k)$ must hold for each $j = 1, \dots, N$ and $k = 1, \dots, N$. Therefore, ℓ'_{OPT} is a lower bound on $P\left(\bigcup_{i=1}^N A_i\right)$. Furthermore, since ℓ'_{OPT} is established in terms of only $\{\alpha_i\}$'s and $\{\gamma_i\}$'s, we have $\ell'_{\text{OPT}} \in \mathcal{L}$. Thus, we only need to prove $\ell'_{\text{OPT}} \geq \ell'$ for all $\ell' \in \mathcal{L}$. Also, note that for the solution of (6), since $\ell'_{\text{OPT}} \leq P\left(\bigcup_{i=1}^N A_i\right) \leq 1$, the objective value must be no larger than 1, i.e., $\sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \leq 1$. Thus, the optimal feasible point of (6) must fall into the subset of the feasible set of (6), which is determined by the additional constraint $\sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \leq 1$. In the following, we prove that ℓ'_{OPT} is achievable, i.e., a family of events $\{A_i\}$ can always be constructed from the solution of (6). Then, the optimality of ℓ'_{OPT} follows by Lemma 1.

Achievability: We prove that for any $\{a_i(k)\}$ that satisfies the constraints of (6) and the additional constraint $\sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \leq 1$, it is always possible to construct a family of events $\{A_i\}$ such that $P(\{\omega \subseteq A_i : \deg(\omega) = k\}) = a_i(k)$ holds. The construction method is given as follows:

- The set Ω' is composed of $N \times N$ atoms, denoted as $\{\omega_i^{(k)}, i = 1, \dots, N, k = 1, \dots, N\}$. In the following, $\{\omega_i^{(k)}, i = 1, \dots, N\}$ are constructed separately for each k .
- Consider N circles such that the k -th circle has a perimeter equals to $\sum_{i=1}^N \frac{a_i(k)}{k}$, $k = 1, \dots, N$. Then for the k -th circle, $\sum_{i=1}^N a_i(k)$ equals

k times its perimeter. Furthermore, since $a_j(k) \leq \sum_{i=1}^N \frac{a_i(k)}{k}$ for all j , $a_j(k)$ is no larger than the perimeter of the k -th circle.

- For $j = 1, \dots, N$, we map the points on the arc of length $a_j(k)$ on the k -th circle from $2\pi \frac{k \sum_{l=1}^{j-1} a_l(k)}{\sum_{i=1}^N a_i(k)}$ to $2\pi \frac{k \sum_{l=1}^j a_l(k)}{\sum_{i=1}^N a_i(k)}$ to a set $B_j^{(k)}$. Then since for the k -th circle, $\sum_{i=1}^N a_i(k)$ equals to k times its perimeter and $a_j(k)$ is no larger than its perimeter, it follows that every point on the k -th circle is mapped to exactly k distinct sets in $\{B_1^{(k)}, \dots, B_N^{(k)}\}$.
- On the k -th circle, the points at the following N angles,

$$2\pi \left(\frac{k \sum_{l=1}^j a_l(k)}{\sum_{i=1}^N a_i(k)} - \left\lfloor \frac{k \sum_{l=1}^j a_l(k)}{\sum_{i=1}^N a_i(k)} \right\rfloor \right), \quad j = 1, \dots, N$$

divide the circle into (at most) N arcs, and the points on each arc are mapped to the same k sets in $\{B_1^{(k)}, \dots, B_N^{(k)}\}$. Let $\{\theta_j^{(k)}, j = 1, \dots, N\}$ be the ordered tuple of

$$\left\{ 2\pi \left(\frac{k \sum_{l=1}^j a_l(k)}{\sum_{i=1}^N a_i(k)} - \left\lfloor \frac{k \sum_{l=1}^j a_l(k)}{\sum_{i=1}^N a_i(k)} \right\rfloor \right), j = 1, \dots, N \right\},$$

then $0 = \theta_1^{(k)} \leq \theta_2^{(k)} \leq \dots \leq \theta_N^{(k)} \leq 2\pi$. Construct the atom $\omega_j^{(k)}$ such that its probability $p(\omega_j^{(k)})$ equals to the length of the j -th arc of the k -th circle, i.e.,

$$p(\omega_j^{(k)}) = \begin{cases} (\theta_{j+1}^{(k)} - \theta_j^{(k)}) \frac{\sum_{i=1}^N a_i(k)}{2\pi k} & \text{for } j < N, \\ (2\pi - \theta_N^{(k)}) \frac{\sum_{i=1}^N a_i(k)}{2\pi k} & \text{for } j = N. \end{cases} \quad (7)$$

- Since the points on the j -th arc are mapped to $\{B_{i_{1j}}^{(k)}, \dots, B_{i_{kj}}^{(k)}\}$ where $\{i_{1j}, \dots, i_{kj}\} \in \{1, \dots, N\}$ that contains k different numbers, we let the atom $\omega_j^{(k)}$ is a subset of $A_{i_{1j}}, \dots, A_{i_{kj}}$, respectively, i.e., $\omega_j^{(k)} \subseteq A_{i_{1j}} \cap \dots \cap A_{i_{kj}}$.
- For each k , the total probability of all constructed atoms equals to the perimeter of the circle, $\sum_{i=1}^N \frac{a_i(k)}{k}$. Also, each atom $\omega_j^{(k)}$ is contains in exactly k events of A_1, \dots, A_N . Finally, since there are in total $N \times N$ atoms $\{\omega_j^{(k)}, j = 1, \dots, N, k = 1, \dots, N\}$, each constructed A_i contains a finite number of atoms.

With the construction described above, it can be readily checked that the constructed $\{A_i\}$ satisfy $P(\{\omega \subseteq A_i : \deg(\omega) = k\}) = a_i(k)$ for all $i = 1, \dots, N$. Since ℓ'_{OPT} is achieved at one feasible point of (6), by the proposed construction method a family of events, say $\{A_i^*\}$, can be constructed so that $P\left(\bigcup_{i=1}^N A_i^*\right) = \ell'_{\text{OPT}}$. Since the first two constraints of (6) are also satisfied, we have $P(A_i^*) = \alpha_i$ and $\sum_j P(A_i^* \cap A_j^*) = \gamma_i$ for all i .

Finally, the optimality of ℓ'_{OPT} directly follows by Lemma 1. ■

Example 2. We give an example in a finite probability space to illustrate the construction provided in the achievability part of the above proof for $N = 4$ and $k = 2$. Assume that $a_1(k) = 0.1$, $a_2(k) = 0.2$, $a_3(k) = 0.3$, and $a_4(k) = 0.4$ for $k = 2$. Since $a_j(k) \leq \sum_{i=1}^4 \frac{a_i(k)}{k} = 0.5$ hold for $j = 1, \dots, 4$, the given $a_j(k)$'s satisfy the constraints in (6) for $k = 2$.

- In order to construct the outcomes, we assume there is a circle with perimeter equals to $\sum_{i=1}^4 \frac{a_i(k)}{k} = 0.5$. Then we map the arc $(0, 0.4\pi)$ to $B_1^{(2)}$, $(0.4\pi, 1.2\pi)$ to $B_2^{(2)}$, $(1.2\pi, 2.4\pi)$ to $B_3^{(2)}$, and $(2.4\pi, 4\pi)$ to $B_4^{(2)}$, as shown in Fig. 1. Then every arc generates an angle less than 2π and every point on the circle is mapped to exactly two sets in $\{B_1^{(2)}, B_2^{(2)}, B_3^{(2)}, B_4^{(2)}\}$. That is: the arc $(0, 0.4\pi)$ is mapped to $B_1^{(2)}$ and $B_3^{(2)}$; the arc $(0.4\pi, 1.2\pi)$ is mapped to $B_2^{(2)}$ and $B_4^{(2)}$; the arc $(1.2\pi, 2\pi)$ is mapped to $B_3^{(2)}$ and $B_4^{(2)}$.
- Since the ordered tuple of the angles $\{0.4\pi, 1.2\pi, 2\pi(1.2 - 1), 2\pi(2 - 2)\}$ is $\{0, 0.4\pi, 0.4\pi, 1.2\pi\}$, the circle is divided by $N = 4$ arcs with lengths equal to $\{0.1, 0, 0.2, 0.2\}$, respectively.
- The outcomes $\omega_1^{(2)}$, $\omega_2^{(2)}$, $\omega_3^{(2)}$ and $\omega_4^{(2)}$ are constructed with probabilities equal to the length of the arcs, i.e., $p(\omega_1^{(2)}) = 0.1$, $p(\omega_2^{(2)}) = 0$, $p(\omega_3^{(2)}) = 0.2$, $p(\omega_4^{(2)}) = 0.2$. Finally, we set the outcomes belonging to events A_i 's as follows: $\omega_1^{(2)} \in A_1 \cap A_3$, $\omega_3^{(2)} \in A_2 \cap A_4$ and $\omega_4^{(2)} \in A_3 \cap A_4$. After the construction for $k = 2$ only, the events of $\{A_i\}$ become: $A_1 = \{\omega_1^{(2)}\}$, $A_2 = \{\omega_3^{(2)}\}$, $A_3 = \{\omega_1^{(2)}, \omega_4^{(2)}\}$, $A_4 = \{\omega_3^{(2)}, \omega_4^{(2)}\}$. Thus, $P(\{\omega \in A_i : \deg(\omega) = k\}) = a_i(k)$ is satisfied for $i = 1, \dots, 4$ and $k = 2$.

Remark 2. Though the number of variables in the optimal lower bound (6) is N^2 , one can further reduce it to $N^2 - N + 1$ by observing that for $k = N$

the constraint $\sum_{i=1}^N a_i(k) \geq ka_j(k)$ is equivalent to $a_1(N) = a_2(N) = \dots = a_N(N)$. Thus, the N variables $a_i(N), i = 1, \dots, N$ can be replaced by only one. The number variables of the optimal lower bound (6) is then $N^2 - N + 1$.

Remark 3. (Optimal Numerical Upper Bound) Since we have proved any point in the subset of the feasible set of (6), determined by $\sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \leq 1$, is achievable, maximizing the objective function in (6), instead of minimizing it, with the same constraints in (6) as well as the additional constraint $\sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \leq 1$, we can obtain an optimal upper bound in the class of all upper bounds that are expressed in terms of only the $P(A_i)$'s and the $\sum_j P(A_i \cap A_j)$'s. For the maximization problem, the additional constraint $\sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \leq 1$ is to ensure that the obtained upper bound is no larger than 1.

Remark 4. Note that in the proof of achievability of Theorem 1, only the last two constraints of (6) with the additional constraint $\sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \leq 1$ are required. Therefore, in other cases where different information is available, optimal lower/upper bounds can be obtained using the same methodology of Theorem 1. For example, in the classes of lower and upper bounds which are established in terms of only $P(A_i) = \alpha_i, i = 1, \dots, N$, the optimal lower and upper bounds in these classes can be obtained by the following problems:

$$\begin{aligned}
& \min_{\{a_i(k)\}} / \max_{\{a_i(k)\}} \quad \sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \\
& \text{s.t.} \quad \sum_{k=1}^N a_i(k) = \alpha_i, \quad i = 1, \dots, N, \\
& \quad \sum_{i=1}^N a_i(k) \geq ka_j(k), \quad j = 1, \dots, N, \quad k = 1, \dots, N, \\
& \quad a_i(k) \geq 0, \quad i = 1, \dots, N, \quad k = 1, \dots, N, \\
& \quad \sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \leq 1.
\end{aligned} \tag{8}$$

Note that the additional constraint $\sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \leq 1$ can be relaxed for the minimization problem, since it is redundant. For the maximization problem, however, it can be active, in which case the obtained upper bound is equal

to 1. Solving the LP problems of (8), we get that the optimal lower bound $P\left(\bigcup_{i=1}^N A_i\right) \geq \max_i \alpha_i$, and that the optimal upper bound $P\left(\bigcup_{i=1}^N A_i\right) \leq \min\{\sum_i P(A_i), 1\}$, i.e., the minimum of 1 and the union upper bound. The optimality of these bounds can be proved using similar arguments as the proof of Theorem 1.

Remark 5. The existing DS bound [6] is known to be optimal in the class of lower bounds with the information $\theta = (S_1 = \sum_{i=1}^N P(A_i), S_2 = \sum_{i,j} P(A_i \cap A_j))$ (e.g., see [5, p.22]). Using Lemma 1, we herein provide a different proof of the optimality of the DS bound [6]. Specifically, we only need to show that the DS bound is the solution of the following LP problem:

$$\begin{aligned}
& \min_{\{a_i(k)\}} \sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \\
& \text{s.t.} \quad \sum_{i=1}^N \sum_{k=1}^N a_i(k) = S_1, \quad \sum_{i=1}^N \sum_{k=1}^N k a_i(k) = S_2, \\
& \quad \sum_{i=1}^N a_i(k) \geq k a_j(k), \quad j = 1, \dots, N, \quad k = 1, \dots, N, \\
& \quad a_i(k) \geq 0, \quad i = 1, \dots, N, \quad k = 1, \dots, N.
\end{aligned} \tag{9}$$

The last two constraints in (9) together with $\sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \leq 1$ guarantee the achievability of the solution of (9). Thus, by Lemma 1, the solution of (9) is the optimal lower bound.

Defining $a(k) = \sum_{i=1}^N a_i(k)$, we show that the problem (9) has the same objective value as the following relaxed LP problem:

$$\begin{aligned}
& \min_{\{a(k)\}} \sum_{k=1}^N \frac{a(k)}{k} \\
& \text{s.t.} \quad \sum_{k=1}^N a(k) = S_1, \quad \sum_{k=1}^N k a(k) = S_2, \\
& \quad a(k) \geq 0, \quad i = 1, \dots, N, \quad k = 1, \dots, N.
\end{aligned} \tag{10}$$

Since (10) is a relaxed problem of (9), it suffices to show that from the solution of (10), we can find a feasible point of (9) that gives the same objective

value. This is done by setting $a_j(k) = a(k)/N$ for all $j = 1, \dots, N$, since all the relaxed constraints $a(k) = \sum_{i=1}^N a_i(k) \geq ka_j(k)$ and $a_i(k) \geq 0$ must be satisfied. Finally, note that the problem (10) is the same as the problem solved in [6]. Therefore, the solution of (10) is the DS bound.

2.2. New Analytical Lower Bound

In this subsection, we derive a new analytical lower bound, which is given in the following theorem.

Theorem 2 (New Analytical Bound). *The lower bound is given by*

$$P\left(\bigcup_{i=1}^N A_i\right) \geq \ell_{NEW} := \delta + \sum_{i=1}^N \left\{ \left[\frac{1}{\chi(\frac{\gamma'_i}{\alpha'_i})} - \frac{\frac{\gamma'_i}{\alpha'_i} - \chi(\frac{\gamma'_i}{\alpha'_i})}{[1 + \chi(\frac{\gamma'_i}{\alpha'_i})][\chi(\frac{\gamma'_i}{\alpha'_i})]} \right] \alpha'_i \right\}, \quad (11)$$

where the function $\chi(\cdot)$ is defined by

$$\chi(x) = \begin{cases} n-1 & \text{if } x = n \text{ where } n \geq 2 \text{ is a integer} \\ \lfloor x \rfloor & \text{otherwise} \end{cases} \quad (12)$$

and

$$\delta := \left\{ \max_i [\gamma_i - (N-1)\alpha_i] \right\}^+ \geq 0, \quad \alpha'_i := \alpha_i - \delta, \quad \gamma'_i := \gamma_i - N\delta. \quad (13)$$

Proof. The new lower bound is the solution of the following relaxed LP:

$$\begin{aligned} \min_{\{a_i(k), i=1, \dots, N, k=1, \dots, N\}} & \sum_{i=1}^N \sum_{k=1}^N \frac{a_i(k)}{k} \\ \text{s.t.} & \sum_{k=1}^N a_i(k) = \alpha_i, \quad \sum_{k=1}^N ka_i(k) = \gamma_i, \quad i = 1, \dots, N, \\ & \sum_{i=1}^N a_i(N) \geq Na_j(N), \quad j = 1, \dots, N \\ & a_i(k) \geq 0, \quad i = 1, \dots, N, \quad k = 1, \dots, N. \end{aligned} \quad (14)$$

Note that the above problem is a relaxed problem of (6) because the constraints $\sum_{i=1}^N a_i(k) \geq ka_j(k)$, $j = 1, \dots, N$ for all $k \neq N$ in (6) are relaxed.

Comparing with the LP problem of (4) that corresponds to the KAT bound, the additional constraints are only $\sum_{i=1}^N a_i(N) \geq Na_j(N), j = 1, \dots, N$, which can be easily proved to be equivalent to requiring that $a_1(N) = a_2(N) = \dots = a_N(N)$. We first introduce a new non-negative variable $x := a_1(N) = \dots = a_N(N)$, and solve the problem (14) by assuming that x is known. Then the objective function in (14) becomes a function of x . Finally, we minimize the objective function to yield a solution of (14).

Replacing $a_i(N), i = 1, \dots, N$ in (14) by x and assuming that x is given implies that (14) can be solved separately for each $i, i = 1, \dots, N$, by solving the following N problems:

$$\begin{aligned}
f_i(x) := \min_{a_i(k), k=1, \dots, N-1} & \sum_{k=1}^{N-1} \frac{a_i(k)}{k} + \frac{x}{N} \\
\text{s.t.} & \sum_{k=1}^{N-1} a_i(k) = \alpha_i - x \\
& \sum_{k=1}^{N-1} ka_i(k) = \gamma_i - Nx \\
& a_i(k) \geq 0, k = 1, \dots, N-1.
\end{aligned} \tag{15}$$

Note that when x is given the above problem is equivalent to (4), the solution of which was derived in different ways in [1, 19]. However, since x is a variable which is assumed to be fixed at the current stage, the solution of problem (15) may not exist for any given x . Thus, one needs to investigate the condition for the existence of a solution for (15) when solving it. To this end, we solve the problem (15) by taking into account the feasible set for x .

Since the LP problem (15) has $N-1$ variables and the LP optimum must be achieved at one of vertices of the polyhedron formed by the constraints [20], the $N-3$ of the $N-1$ constraints $a_i(k) \geq 0, k = 1, \dots, N-1$ must be active. Assume that the other two constraints $a_i(k) \geq 0$ that are not active are given for $k = k_1$ and $k = k_2$ and $1 \leq k_1 < k_2 \leq N-1$, then we obtain

$$a_i(k_1) + a_i(k_2) = \alpha_i - x, \quad k_1 a_i(k_1) + k_2 a_i(k_2) = \gamma_i - Nx, \tag{16}$$

which yields

$$a_i(k_1) = \frac{k_2(\alpha_i - x) - (\gamma_i - Nx)}{k_2 - k_1} \geq 0, \quad a_i(k_2) = \frac{(\gamma_i - Nx) - k_1(\alpha_i - x)}{k_2 - k_1} \geq 0. \tag{17}$$

Using the condition $1 \leq k_1 < k_2 \leq N - 1$, the solution exists when

$$[\gamma_i - (N - 1)\alpha_i]^+ \leq x \leq \frac{\beta_i}{N - 1}, \quad (18)$$

and k_2 and k_1 satisfy $k_1 \leq \frac{\gamma_i - Nx}{\alpha_i - x} \leq k_2$.

Since it is easy to prove that $\frac{a_i(k_1)}{k_1} + \frac{a_i(k_2)}{k_2}$ is non-decreasing with k_2 and non-increasing with k_1 (see Appendix B), the optimal k_2 and k_1 when $\frac{\gamma_i - Nx}{\alpha_i - x}$ is not an integer are

$$k_1 = \left\lfloor \frac{\gamma_i - Nx}{\alpha_i - x} \right\rfloor, \quad k_2 = k_1 + 1. \quad (19)$$

When $\frac{\gamma_i - Nx}{\alpha_i - x}$ is an integer, one can choose either $k_1 = \frac{\gamma_i - Nx}{\alpha_i - x}, k_2 = k_1 + 1$ or $k_1 = \frac{\gamma_i - Nx}{\alpha_i - x} - 1, k_2 = k_1 + 1$, since for both cases the values of $\frac{a_i(k_1)}{k_1} + \frac{a_i(k_2)}{k_2}$ are indeed identical. Note that the condition for the existence of the solution to (14) implies that $1 \leq k_1 < k_2 \leq N - 1$, thus, the optimal k_1 and k_2 that give the largest feasible set of x are

$$k_1 = \chi\left(\frac{\gamma_i - Nx}{\alpha_i - x}\right), \quad k_2 = k_1 + 1. \quad (20)$$

Then the solution of (15) which is a function of x can be written as

$$\begin{aligned} f_i(x) = & \frac{2\chi\left(\frac{\gamma_i - Nx}{\alpha_i - x}\right) + 1}{\chi\left(\frac{\gamma_i - Nx}{\alpha_i - x}\right) \left[\chi\left(\frac{\gamma_i - Nx}{\alpha_i - x}\right) + 1\right]} (\alpha_i - x) \\ & - \frac{1}{\chi\left(\frac{\gamma_i - Nx}{\alpha_i - x}\right) \left[\chi\left(\frac{\gamma_i - Nx}{\alpha_i - x}\right) + 1\right]} (\gamma_i - Nx) + \frac{x}{N}, \end{aligned} \quad (21)$$

where $[\gamma_i - (N - 1)\alpha_i]^+ \leq x \leq \frac{\beta_i}{N - 1}$.

Next, we prove that $f_i(x)$ is a non-decreasing function of x . First, we prove that the function $f_i(x)$ is continuous. Note that $\frac{\gamma_i - Nx}{\alpha_i - x}$ is strictly decreasing with x if $\frac{\gamma_i}{\alpha_i} < N$ (see Appendix C). When $\frac{\gamma_i - Nx}{\alpha_i - x}$ is an integer, say $\frac{\gamma_i - Nx}{\alpha_i - x} = n \leq N - 1$, choose $h > 0$ satisfies $n - 1 < \frac{\gamma_i - N(x+h)}{\alpha_i - (x+h)} < n$ and $n < \frac{\gamma_i - N(x-h)}{\alpha_i - (x-h)} < n + 1$. Then we have $\chi\left(\frac{\gamma_i - Nx}{\alpha_i - x}\right) = n - 1$, $\chi\left(\frac{\gamma_i - N(x+h)}{\alpha_i - (x+h)}\right) = n - 1$ and $\chi\left(\frac{\gamma_i - N(x-h)}{\alpha_i - (x-h)}\right) = n$. Then $f_i(x + h) - f_i(x) = \left(\frac{1}{n-1} - \frac{1}{N}\right) \frac{N-n}{n} h > 0$ and $f_i(x) - f_i(x - h) = \left(\frac{1}{n+1} - \frac{1}{N}\right) \frac{N-n}{n} h \geq 0$ (see Appendix D). Both

$f_i(x+h) - f_i(x)$ and $f_i(x) - f_i(x-h)$ tend to zero when $h \rightarrow 0$. Thus, the function $f_i(x)$ is continuous when $\frac{\gamma_i - Nx}{\alpha_i - x}$ is an integer.

When $\frac{\gamma_i - Nx}{\alpha_i - x}$ is not an integer, $\chi(\frac{\gamma_i - Nx}{\alpha_i - x}) \leq N - 1$ and the function $f_i(x)$ is continuous and differentiable. The derivative of $f_i(x)$ satisfies

$$\begin{aligned} f'_i(x) &= \frac{1}{N} - \frac{1}{\chi(\frac{\gamma_i - Nx}{\alpha_i - x})} - \frac{1}{\chi(\frac{\gamma_i - Nx}{\alpha_i - x}) + 1} + \frac{N}{\chi(\frac{\gamma_i - Nx}{\alpha_i - x}) \left[\chi(\frac{\gamma_i - Nx}{\alpha_i - x}) + 1 \right]} \\ &= \frac{\left[N - \chi(\frac{\gamma_i - Nx}{\alpha_i - x}) \right] \left[N - \chi(\frac{\gamma_i - Nx}{\alpha_i - x}) - 1 \right]}{N \chi(\frac{\gamma_i - Nx}{\alpha_i - x}) \left[\chi(\frac{\gamma_i - Nx}{\alpha_i - x}) + 1 \right]} \geq 0, \end{aligned} \quad (22)$$

which means that $f_i(x)$ is a non-decreasing function of x . Then we can finally solve the problem (14) to get the new lower bound

$$\ell_{\text{NEW}} = \min_x \left[\sum_{i=1}^N f_i(x) \right] \quad \text{s.t.} \quad \left\{ \max_i [\gamma_i - (N-1)\alpha_i] \right\}^+ \leq x \leq \min_i \frac{\beta_i}{N-1}, \quad (23)$$

where ℓ_{NEW} denotes the new analytical bound. Since $\sum_{i=1}^N f_i(x)$ is non-decreasing in x , defining $\delta = \left\{ \max_i [\gamma_i - (N-1)\alpha_i] \right\}^+$, the objective value is thus obtained at $x = \delta$ so that $\ell_{\text{NEW}} = \sum_{i=1}^N f_i(\delta)$. ■

Remark 6. We can also prove that the LP problem (14) has a unique optimal feasible point. Note that for a given x , the problem (14) reduces to the LP problem (4) for the KAT bound, which has a unique optimal feasible point by Remark 1. Therefore, it suffices to show that there does not exist $\delta < \delta^* \leq \min_i \frac{\beta_i}{N-1}$ such that $\sum_{i=1}^N f_i(\delta^*) = \sum_{i=1}^N f_i(\delta)$. Since $f_i(x)$ is non-negative and non-decreasing with x , it suffices to show that there does not exist $\delta < \delta^* \leq \min_i \frac{\beta_i}{N-1}$ such that $f'_i(\delta^*) = 0$ for all i . From (22), we know that $f'_i(\delta^*) = 0$ if and only if $\chi(\frac{\gamma_i - N\delta^*}{\alpha_i - \delta^*}) = N - 1$, which is equivalent to $N - 1 < \frac{\gamma_i - N\delta^*}{\alpha_i - \delta^*} \leq N$. In the following, we prove the nonexistence of δ^* by considering three cases: (i) If $\delta = 0$ for some i , which means $\frac{\gamma_i}{\alpha_i} \leq N - 1$, then $\frac{\gamma_i - N\delta^*}{\alpha_i - \delta^*} \leq \frac{\gamma_i}{\alpha_i} \leq N - 1$, which is a contradiction with $N - 1 < \frac{\gamma_i - N\delta^*}{\alpha_i - \delta^*} \leq N$; (ii) If $\delta > 0$ and $\frac{\gamma_i}{\alpha_i} = N$ for some i , then $x = \alpha_i = \delta$ is the only point in the feasible set of (23), thus $\delta^* > \delta$ does not exist; (iii) If $\delta > 0$ and $\frac{\gamma_i}{\alpha_i} < N$ for all i , then $\delta = \max_i [\gamma_i - (N-1)\alpha_i] = \gamma_k - (N-1)\alpha_k$ for some $k \in \{1, \dots, N\}$. Since $\frac{\gamma_k - Nx}{\alpha_k - x}$ strictly decreases with x , we have

$\frac{\gamma_k - N\delta^*}{\alpha_k - \delta^*} < \frac{\gamma_k - N\delta}{\alpha_k - \delta} = \frac{\gamma_k - N[\gamma_k - (N-1)\alpha_k]}{\alpha_k - [\gamma_k - (N-1)\alpha_k]} = \frac{(N-1)(N\alpha_k - \gamma_k)}{N\alpha_k - \gamma_k} = N - 1$, which contradicts with $\frac{\gamma_k - N\delta^*}{\alpha_k - \delta^*} > N - 1$. Thus, the optimal feasible point of (23) is unique so that the optimal feasible point of (14) is unique. Therefore, the new analytical bound is achievable if and only if the optimal feasible point of the LP problem (14) has a corresponding family of events $\{A_i, i = 1, \dots, N\}$ that satisfies the information represented by $\theta = (\alpha_1, \dots, \alpha_N, \gamma_1, \dots, \gamma_N)$.

Remark 7. It can be easily seen by comparing the LP problems of (4) and (14) that the new analytical bound is at least as good as the KAT bound. This is because the feasible set of (4) contains the feasible set of (14), and both problems (4) and (14) share the same objective function. Furthermore, setting $\delta = 0$ directly yields $\ell_{\text{NEW}} = \ell_{\text{KAT}}$.

3. Comparison of the new analytical bound with the KAT Bound

In this section, we give a lower bound on $\ell_{\text{NEW}} - \ell_{\text{KAT}}$ which is given in the following lemma.

Lemma 3. A lower bound on $\ell_{\text{NEW}} - \ell_{\text{KAT}}$ is given as follows:

$$\ell_{\text{NEW}} - \ell_{\text{KAT}} \geq \left\{ \sum_{i=1}^N \frac{\left[N - \chi\left(\frac{\gamma_i}{\alpha_i}\right) \right] \left[N - \chi\left(\frac{\gamma_i}{\alpha_i}\right) - 1 \right]}{\chi\left(\frac{\gamma_i}{\alpha_i}\right) \left[\chi\left(\frac{\gamma_i}{\alpha_i}\right) + 1 \right]} \right\} \frac{\delta}{N}, \quad (24)$$

where the strict inequality holds if and only if there exists $0 < \delta' < \delta$ that $\frac{\gamma_i - N\delta'}{\alpha_i - \delta'}$ is an integer for some $i \in \{1, \dots, N\}$.

Proof. We first prove that $f_i(x)$ is convex in x . Note that $f_i(x)$ is a continuous and piecewise differentiable function. However, it is not differentiable when $\frac{\gamma_i - Nx}{\alpha_i - x}$ is an integer. In each interval of x where $\frac{\gamma_i - Nx}{\alpha_i - x}$ is between two successive integers, the derivative of $f_i(x)$ is given by (22) which is positive and only a function of $\chi\left(\frac{\gamma_i - Nx}{\alpha_i - x}\right)$. Since $\chi\left(\frac{\gamma_i - Nx}{\alpha_i - x}\right)$ is an integer that does not change in each interval where $\frac{\gamma_i - Nx}{\alpha_i - x}$ is between two successive integers, we only need to show that the derivative of $f_i(x)$ given by (22) is a non-decreasing function of x . By denoting $n(x) := \chi\left(\frac{\gamma_i - Nx}{\alpha_i - x}\right)$, we can write $f'_i(x) = g_i(n)$ where

$$g_i(n) := \frac{(N - n)(N - n - 1)}{N(n + 1)n}. \quad (25)$$

Noting that $\gamma_i \leq N\alpha_i$, one can verify that $\frac{\gamma_i - Nx}{\alpha_i - x}$ decreases with x and by the definition of $\chi(\cdot)$, $n \leq N - 1$ and $n = \chi(\frac{\gamma_i - Nx}{\alpha_i - x})$ is a non-increasing function of x . Thus, we have $g_i(n) > 0$ and $g_i(n)$ is a decreasing function of n for $1 < n \leq N - 1$, since

$$g_i(n) - g_i(n-1) = \frac{(N-n)(N-n-1)}{N(n+1)n} - \frac{(N-n+1)(N-n)}{Nn(n-1)} < 0, \quad (26)$$

which implies $f'_i(x)$ is a non-decreasing function of x . Therefore $f_i(x)$ is a convex function of x . Finally, by the property of a convex function, we have

$$f_i(x) - f_i(0) \geq f'_i(0)(x - 0). \quad (27)$$

Since $\ell_{\text{NEW}} - \ell_{\text{KAT}} = \sum_i [f_i(\delta) - f_i(0)]$, by substituting $x = \delta$ into (27) and summing over i , the inequality of (24) is obtained.

Note that if for all $i = 1, \dots, N$ there does not exist $0 < \delta' < \delta$ such that $\frac{\gamma_i - N\delta'}{\alpha_i - \delta'}$ is an integer, the derivative $f'_i(x) = f'_i(0)$ for all $0 < x < \delta$ and $i = 1, \dots, N$. Then equality in (27) holds for all i , and equality holds in (24). If there exists $0 < \delta' < \delta$ such that $\frac{\gamma_i - N\delta'}{\alpha_i - \delta'}$ is an integer for some i , then according to (26) and the definition of $\chi(\cdot)$, we have $f'_i(\delta') > f'_i(0)$. Then, it can be shown that for those i the strict inequality in (27) holds when $x = \delta$. This is because

$$\begin{aligned} f_i(\delta) - f_i(0) &= [f_i(\delta) - f_i(\delta')] + [f_i(\delta') - f_i(0)] \\ &\geq f'_i(\delta')(\delta - \delta') + f'_i(0)(\delta' - 0) \\ &> f'_i(0)(\delta - \delta') + f'_i(0)(\delta' - 0) \\ &= f'_i(0)(\delta - 0). \end{aligned} \quad (28)$$

Therefore, the strictly inequality in (24) holds. ■

Remark 8. Lemma 3 readily yields another analytical lower bound, which is sharper than the KAT bound but looser than the new analytical bound:

$$P\left(\bigcup_{i=1}^N A_i\right) \geq \ell_{\text{KAT}} + \left\{ \sum_{i=1}^N \frac{\left[N - \chi\left(\frac{\gamma_i}{\alpha_i}\right)\right] \left[N - \chi\left(\frac{\gamma_i}{\alpha_i}\right) - 1\right]}{\chi\left(\frac{\gamma_i}{\alpha_i}\right) \left[\chi\left(\frac{\gamma_i}{\alpha_i}\right) + 1\right]} \right\} \frac{\delta}{N}. \quad (29)$$

We herein show that the above analytical lower bound can be derived directly based on the arguments in [1].

Recall [1, the last equation above (8)] that

$$V_i := \sum_{k=1}^N \frac{a_i(k)}{k} = \frac{2}{r} \alpha_i - \frac{1}{r(r-1)} \beta_i + \sum_{k=1}^N \frac{(r-k)(r-k-1)}{r(r-1)} \frac{a_i(k)}{k}, \quad (30)$$

where the last term

$$\sum_{k=1}^N \frac{(r-k)(r-k-1)}{r(r-1)} \frac{a_i(k)}{k} \geq 0. \quad (31)$$

Thus, the authors in [1] use

$$V_i \geq \frac{2}{r} \alpha_i - \frac{1}{r(r-1)} \beta_i \quad (32)$$

to determine the KAT bound as follows

$$P \left(\bigcup_i A_i \right) = \sum_i V_i \geq \max_{r_i \in \mathbb{N}} \sum_i \left[\frac{2}{r_i} \alpha_i - \frac{1}{r_i(r_i-1)} \beta_i \right] = \ell_{KAT}, \quad (33)$$

where the maximization is over the integer r_i 's and the optimum is achieved at $r_i = 1 + \chi(\frac{2\alpha_i}{\alpha_i})$. Note that we can rewrite (30) as

$$\begin{aligned} V_i = & \left[\frac{2}{r} \alpha_i - \frac{1}{r(r-1)} \beta_i + \frac{(r-N)(r-N-1)}{r(r-1)} \frac{a_i(N)}{N} \right] \\ & + \sum_{k=1}^{N-1} \frac{(r-k)(r-k-1)}{r(r-1)} \frac{a_i(k)}{k}, \end{aligned} \quad (34)$$

where the last term is non-negative. Note that from (23), one can see that δ is the smallest possible value of $x = a_i(N)$ that guarantees the existence of the solution of (14), i.e. $\frac{\delta}{N} \leq \frac{a_i(N)}{N}$. Therefore, the lower bound of (29) can

be obtained as follows:

$$\begin{aligned}
P\left(\bigcup_i A_i\right) &= \sum_i V_i \\
&\geq \max_{r_i \in \mathbb{N}} \sum_i \left[\frac{2}{r_i} \alpha_i - \frac{1}{r_i(r_i - 1)} \beta_i + \frac{(r_i - N)(r_i - N - 1)}{r_i(r_i - 1)} \frac{a_i(N)}{N} \right] \\
&\geq \sum_i \left\{ \frac{2\alpha_i}{1 + \chi(\frac{\gamma_i}{\alpha_i})} - \frac{\beta_i}{[1 + \chi(\frac{\gamma_i}{\alpha_i})]\chi(\frac{\gamma_i}{\alpha_i})} + \frac{[1 + \chi(\frac{\gamma_i}{\alpha_i}) - N](\chi(\frac{\gamma_i}{\alpha_i}) - N)}{[1 + \chi(\frac{\gamma_i}{\alpha_i})]\chi(\frac{\gamma_i}{\alpha_i})} \frac{\delta}{N} \right\} \\
&= \ell_{KAT} + \left\{ \sum_{i=1}^N \frac{[N - \chi(\frac{\gamma_i}{\alpha_i})][N - \chi(\frac{\gamma_i}{\alpha_i}) - 1]}{\chi(\frac{\gamma_i}{\alpha_i})[\chi(\frac{\gamma_i}{\alpha_i}) + 1]} \right\} \frac{\delta}{N}.
\end{aligned} \tag{35}$$

4. Numerical Examples

In this section, we evaluate the new lower bounds using eight numerical examples. The first four examples are the same as in [1]. The last four examples, Systems V to VIII, are new and are shown in Table 1-4, respectively. As a reference, the existing DS bound [6], de Caen's bound [7], the KAT bound (5) are included for comparison. Furthermore, the GK bound [14, 15], which exploits full information of all $P(A_i \cap A_j)$'s and $P(A_i)$'s, is also compared with the new bounds. The results are shown in Table 5. The gap of $\ell_{\text{NEW}} - \ell_{\text{KAT}}$ and the derived lower bound (24) are shown in Table 6.

One can see that the KAT bound is at least as good as the DS and de Caen's bounds as already shown in [1]. The new bounds are at least as good as the KAT bound in all the examples, as expected. More specifically, the new numerical bound (6) is sharper than the KAT bound in all examples, and the new analytical bound (11) is sharper than the KAT bound for Systems V to VIII and identical to the KAT bound for Systems I to VI. Concerning the gap of new analytical bound and the KAT bound, the equality of (24) holds for Systems V, VII and VIII.

Moreover, from the numerical examples, we note that the GK bound [14, 15], which requires more information (all the information of individual $P(A_i \cap A_j)$ as well as $P(A_i)$'s), is not guaranteed to be sharper than the KAT

bound³ and the new bounds. For example, the GK bound is worse than the KAT bound as well as the new bounds in Systems V and VIII. It is better than the KAT bound but worse than the new bounds in System VI, better than the KAT bound and the new analytical bound but worse than the new numerical bound in System VII.

Finally, we note that all lower bounds considered in this paper can be sharpened algorithmically by optimizing over subsets (e.g., see [9, 13, 10]).

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³Another example in which the GK bound is looser than the KAT bound is given in [18].

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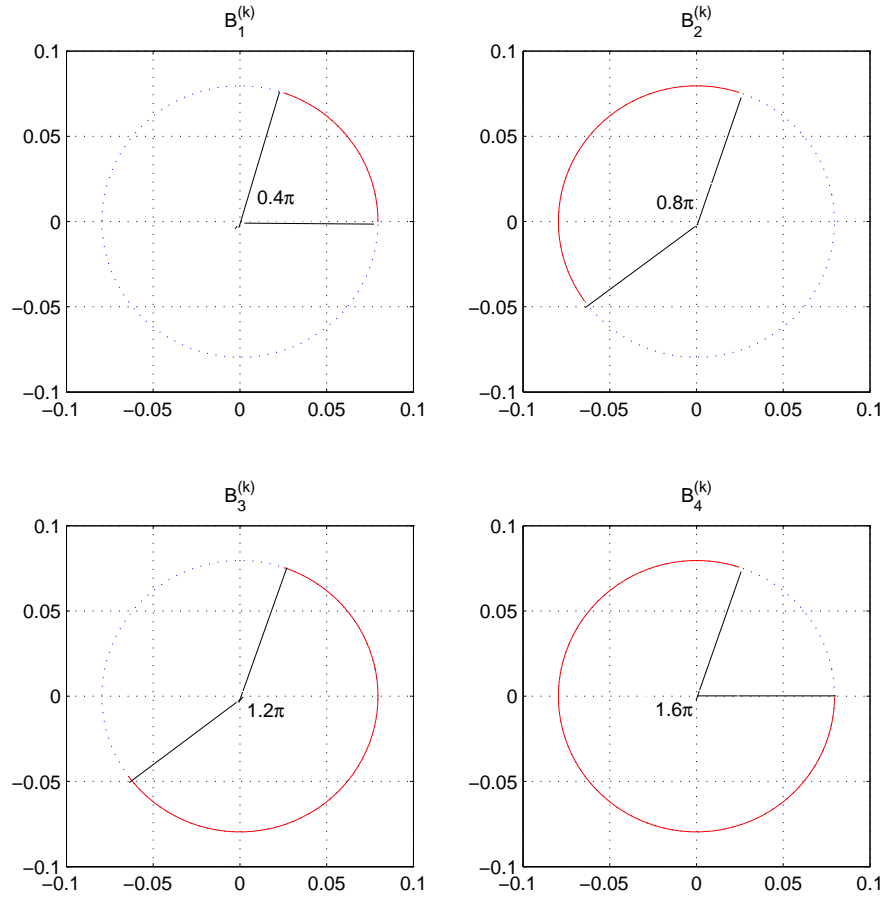


Figure 1: Example illustrating the construction of the proof of Theorem 1 for $N = 4$ and $k = 2$.

Table 1: System V.

Outcomes ω_i	$p(\omega_i)$	A_1	A_2	A_3
ω_0	0.145			\times
ω_1	0.045	\times		\times
ω_2	0.01	\times	\times	\times
ω_3	0.045	\times	\times	
ω_4	0.145		\times	

Table 2: System VI.

Outcomes ω_i	$p(\omega_i)$	A_1	A_2	A_3	A_4
ω_0	0.0962	\times			\times
ω_1	0.0446				\times
ω_2	0.0581		\times		\times
ω_3	0.0225	\times	\times	\times	\times
ω_4	0.0385	\times			
ω_5	0.0071	\times		\times	\times
ω_6	0.0582		\times		

Table 3: System VII.

Outcomes ω_i	$p(\omega_i)$	A_1	A_2	A_3	A_4
ω_0	0.1832	\times			
ω_1	0.1219			\times	
ω_2	0.0337	\times	\times	\times	\times
ω_3	0.0256		\times	\times	
ω_4	0.0682				\times
ω_5	0.0389		\times	\times	\times
ω_6	0.0631			\times	\times

Table 4: System VIII.

Outcomes ω_i	$p(\omega_i)$	A_1	A_2	A_3	A_4
ω_0	0.0330				\times
ω_1	0.0705	\times	\times	\times	
ω_2	0.0876	\times		\times	\times
ω_3	0.0608	\times	\times		\times
ω_4	0.0865	\times	\times	\times	\times
ω_5	0.0621		\times	\times	\times
ω_6	0.0181		\times		
ω_7	0.0898	\times		\times	
ω_8	0.0770		\times	\times	

Table 5: Comparison of Lower Bounds.

System	$P\left(\bigcup_{i=1}^N A_i\right)$	DS	de Caen	KAT	GK	Bound (11)	Bound (6)
I	0.7890	0.7007	0.7087	0.7247	0.7601	0.7247	0.7487
II	0.6740	0.6150	0.6154	0.6227	0.6510	0.6227	0.6398
III	0.7890	0.6933	0.7048	0.7222	0.7508	0.7222	0.7427
IV	0.9687	0.8879	0.8757	0.8909	0.9231	0.8909	0.9044
V	0.3900	0.3800	0.3495	0.3833	0.3813	0.3900	0.3900
VI	0.3252	0.2706	0.2720	0.2769	0.2972	0.3205	0.3252
VII	0.5346	0.3989	0.4186	0.4434	0.4750	0.4562	0.5090
VIII	0.5854	0.5395	0.5352	0.5412	0.5390	0.5464	0.5513

Table 6: Comparison of New Bound (11) with KAT Bound.

System	KAT	Bound (11)	$\ell_{\text{NEW}} - \ell_{\text{KAT}}$	Lower bound on the gap (24)
V	0.3833	0.3900	0.0067	0.0067
VI	0.2769	0.3205	0.0436	0.0206
VII	0.4434	0.4562	0.0128	0.0128
VIII	0.5412	0.5464	0.0051	0.0051

Appendix A. Uniqueness of the optimal feasible point of the KAT LP problem (4)

Since we can solve the KAT LP problem (4) by separating $i = 1, \dots, N$, we only need to consider if the following LP has a unique optimal feasible point:

$$\min_{\mathbf{x}} \mathbf{p}^T \mathbf{x}, \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{I}\mathbf{x} \geq \mathbf{0}, \quad (\text{A.1})$$

where $\mathbf{p} = (1, \frac{1}{2}, \dots, \frac{1}{N})^T$, $\mathbf{x} = (x_1, \dots, x_N)^T$, $\mathbf{b} = (\alpha, \gamma)^T$, \mathbf{I} is the $N \times N$ identity matrix, and

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & N \end{pmatrix}. \quad (\text{A.2})$$

First, the optimum of LP is achieved at one vertex, which is denoted by $\{x_k = 0, k \neq k_1, k \neq k_2\}$ and $x_{k_1} \geq 0, x_{k_2} \geq 0, k_2 > k_1$ such that

$$x_{k_1} + x_{k_2} = \alpha, \quad k_1 x_{k_1} + k_2 x_{k_2} = \gamma, \quad (\text{A.3})$$

or $x_{k_1} = \frac{k_2 \alpha - \gamma}{k_2 - k_1}$ and $x_{k_2} = \frac{\gamma - k_1 \alpha}{k_2 - k_1}$. The optimal k_1^* and k_2^* are chosen to minimize the objective function

$$\{k_1^*, k_2^*\} = \arg \min_{k_1, k_2} \mathbf{p}^T \mathbf{x} = \frac{x_{k_1}}{k_1} + \frac{x_{k_2}}{k_2} \quad (\text{A.4})$$

Denote \mathcal{K} as the set of all integer tuples (k_1, k_2) such that $1 \leq k_1 < k_2 \leq N$, and \mathcal{K}_* be the set of optimum (k_1^*, k_2^*) . Then using a similar approach to the proof of Theorem 2, we can show that if $\frac{\gamma}{\alpha}$ is not an integer, $\mathcal{K}_* = \{(\lfloor \frac{\gamma}{\alpha} \rfloor, \lfloor \frac{\gamma}{\alpha} \rfloor + 1)\}$ has only one optimal solution of (k_1^*, k_2^*) . If $\frac{\gamma}{\alpha}$ is an integer, $\mathcal{K}_* = \{(\frac{\gamma}{\alpha}, \frac{\gamma}{\alpha} + 1), (\frac{\gamma}{\alpha} - 1, \frac{\gamma}{\alpha})\}$, both lead to the same solution of \mathbf{x} . Note that using a similar approach as the one adopted in Appendix B, by computing the derivative with respect to k_1 and k_2 , we have that

$$\frac{x_{k_1^*}}{k_1^*} + \frac{x_{k_2^*}}{k_2^*} = \min_{(k_1, k_2) \in \mathcal{K}} \frac{x_{k_1}}{k_1} + \frac{x_{k_2}}{k_2} < \min_{(k_1, k_2) \in \mathcal{K} \setminus \mathcal{K}_*} \frac{x_{k_1}}{k_1} + \frac{x_{k_2}}{k_2} \quad (\text{A.5})$$

for any $(k_1^*, k_2^*) \in \mathcal{K}_*$. Note that the strict inequality implies there exists $\epsilon^* > 0$ such that

$$\frac{x_{k_1^*}}{k_1^*} + \frac{x_{k_2^*}}{k_2^*} + \epsilon^* \leq \min_{(k_1, k_2) \in \mathcal{K} \setminus \mathcal{K}_*} \frac{x_{k_1}}{k_1} + \frac{x_{k_2}}{k_2}. \quad (\text{A.6})$$

Now we prove the uniqueness of the LP problem by [21, Theorem 1], which is to show that for any given non-zero vector \mathbf{q} , the optimal feasible point

of (A.1) is also optimal for the following perturbed LP problem by properly choosing $\epsilon > 0$:

$$\min_{\mathbf{x}} (\mathbf{p} + \epsilon \mathbf{q})^T \mathbf{x}, \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{I}\mathbf{x} \geq \mathbf{0}. \quad (\text{A.7})$$

Since the constraints remain the same, it suffices to show the optimal (k_1^*, k_2^*) are also optimal for the perturbed problem, i.e.,

$$\left(\frac{1}{k_1^*} + \epsilon q_{k_1^*}\right)x_{k_1^*} + \left(\frac{1}{k_2^*} + \epsilon q_{k_2^*}\right)x_{k_2^*} = \min_{(k_1, k_2) \in \mathcal{K}} \left[\left(\frac{1}{k_1} + \epsilon q_{k_1}\right)x_{k_1} + \left(\frac{1}{k_2} + \epsilon q_{k_2}\right)x_{k_2} \right] \quad (\text{A.8})$$

which is enough to show

$$\left(\frac{1}{k_1^*} + \epsilon q_{k_1^*}\right)x_{k_1^*} + \left(\frac{1}{k_2^*} + \epsilon q_{k_2^*}\right)x_{k_2^*} \leq \min_{(k_1, k_2) \in \mathcal{K} \setminus \mathcal{K}^*} \left[\left(\frac{1}{k_1} + \epsilon q_{k_1}\right)x_{k_1} + \left(\frac{1}{k_2} + \epsilon q_{k_2}\right)x_{k_2} \right] \quad (\text{A.9})$$

Note that

$$\left(\frac{1}{k_1^*} + \epsilon q_{k_1^*}\right)x_{k_1^*} + \left(\frac{1}{k_2^*} + \epsilon q_{k_2^*}\right)x_{k_2^*} \leq \left(\frac{x_{k_1^*}}{k_1^*} + \frac{x_{k_2^*}}{k_2^*}\right) + \epsilon \left[\max_{(k_1, k_2) \in \mathcal{K}} (q_{k_1}x_{k_1} + q_{k_2}x_{k_2}) \right] \quad (\text{A.10})$$

and

$$\begin{aligned} & \min_{(k_1, k_2) \in \mathcal{K} \setminus \mathcal{K}^*} \left[\left(\frac{1}{k_1} + \epsilon q_{k_1}\right)x_{k_1} + \left(\frac{1}{k_2} + \epsilon q_{k_2}\right)x_{k_2} \right] \\ & \geq \min_{(k_1, k_2) \in \mathcal{K} \setminus \mathcal{K}^*} \left(\frac{x_{k_1}}{k_1} + \frac{x_{k_2}}{k_2} \right) + \epsilon \left[\min_{(k_1, k_2) \in \mathcal{K}} (q_{k_1}x_{k_1} + q_{k_2}x_{k_2}) \right] \end{aligned} \quad (\text{A.11})$$

Therefore, by choosing

$$\epsilon = \frac{\epsilon^*}{\max_{(k_1, k_2) \in \mathcal{K}} (q_{k_1}x_{k_1} + q_{k_2}x_{k_2}) - \min_{(k_1, k_2) \in \mathcal{K}} (q_{k_1}x_{k_1} + q_{k_2}x_{k_2})}, \quad (\text{A.12})$$

equation (A.9) can be obtained by (A.10), (A.11) and (A.6).

Appendix B. Proof that $\frac{a_i(k_1)}{k_1} + \frac{a_i(k_2)}{k_2}$ is non-decreasing with k_2 and non-increasing with k_1 (between (18) and (19))

Let $k := \frac{\gamma_i - Nx}{\alpha_i - x}$, then

$$a_i(k_1) = \frac{k_2(\alpha_i - x) - (\gamma_i - Nx)}{k_2 - k_1} = (\alpha_i - x) \frac{k_2 - \frac{\gamma_i - Nx}{\alpha_i - x}}{k_2 - k_1} = (\alpha_i - x) \frac{k_2 - k}{k_2 - k_1}. \quad (\text{B.1})$$

Similarly, we have

$$a_i(k_2) = \frac{(\gamma_i - Nx) - k_1(\alpha_i - x)}{k_2 - k_1} = (\alpha_i - x) \frac{k - k_1}{k_2 - k_1}. \quad (\text{B.2})$$

Since $\alpha_i - x$ is a constant here and $k_1 \leq k \leq k_2$, we only need to consider

$$\begin{aligned} \frac{1}{\alpha_i - x} \left[\frac{a_i(k_1)}{k_1} + \frac{a_i(k_2)}{k_2} \right] &= \frac{1}{k_1} \frac{k_2 - k}{k_2 - k_1} + \frac{1}{k_2} \frac{k - k_1}{k_2 - k_1} \\ &= \frac{1}{k_2 - k_1} \left[\left(\frac{k_2}{k_1} - \frac{k}{k_1} \right) + \left(\frac{k}{k_2} - \frac{k_1}{k_2} \right) \right] \\ &= \frac{1}{k_2 - k_1} \left(\frac{k_2^2 - k_1^2}{k_1 k_2} + \frac{k_1 - k_2}{k_1 k_2} k \right) \\ &= \frac{1}{k_1} + \frac{1}{k_2} - \frac{k}{k_1 k_2} \end{aligned} \quad (\text{B.3})$$

The derivatives w.r.t. k_1 and k_2 can be obtained as follows:

$$\frac{1}{k_1^2} \left(\frac{k}{k_2} - 1 \right) \leq 0, \quad \frac{1}{k_2^2} \left(\frac{k}{k_1} - 1 \right) \geq 0. \quad (\text{B.4})$$

Therefore, the function is non-increasing with k_1 and non-decreasing with k_2 .

Appendix C. Proof that $\frac{\gamma_i - Nx}{\alpha_i - x}$ is a non-increasing function of x

Note that by definition of γ_i and α_i , we know $\gamma_i \leq N\alpha_i$.

$$\begin{aligned} \left(\frac{\gamma_i - Nx}{\alpha_i - x} \right)' &= \frac{(-N)(\alpha_i - x) - (\gamma_i - Nx)(-1)}{(\alpha_i - x)^2} \\ &= \frac{(\gamma_i - Nx) - N(\alpha_i - x)}{(\alpha_i - x)^2} \\ &\leq \frac{(N\alpha_i - Nx) - N(\alpha_i - x)}{(\alpha_i - x)^2} = 0. \end{aligned} \quad (\text{C.1})$$

Clearly, if $\gamma_i < N\alpha_i$, the function $\frac{\gamma_i - Nx}{\alpha_i - x}$ is a strictly decreasing function of x .

Appendix D. Proof that $f_i(x+h) - f_i(x)$ and $f_i(x) - f_i(x-h)$ are positive

By (21), we can write

$$\begin{aligned}
f_i(x) &= \frac{2(n-1)+1}{(n-1)n}(\alpha_i - x) - \frac{1}{(n-1)n}(\gamma_i - Nx) + \frac{x}{N} \\
f_i(x+h) &= \frac{2(n-1)+1}{(n-1)n}(\alpha_i - x - h) - \frac{1}{(n-1)n}(\gamma_i - Nx - Nh) + \frac{x+h}{N} \\
f_i(x-h) &= \frac{2n+1}{n(n+1)}(\alpha_i - x + h) - \frac{1}{n(n+1)}(\gamma_i - Nx + Nh) + \frac{x-h}{N}
\end{aligned} \tag{D.1}$$

Substituting $\gamma_i - Nx = n(\alpha_i - x)$, we get

$$\begin{aligned}
f_i(x) &= \left[\frac{2n-1}{(n-1)n} - \frac{n}{(n-1)n} \right] (\alpha_i - x) + \frac{x}{N} = \frac{1}{n}(\alpha_i - x) + \frac{x}{N} \\
f_i(x+h) &= f_i(x) - \frac{2(n-1)+1}{(n-1)n}h + \frac{N}{(n-1)n}h + \frac{1}{N}h \\
f_i(x-h) &= \left[\frac{2n+1}{n(n+1)} - \frac{n}{n(n+1)} \right] (\alpha_i - x) + \left[\frac{2n+1}{n(n+1)} - \frac{N}{n(n+1)} \right] h + \frac{x-h}{N} \\
&= f_i(x) + \left[\frac{2n+1}{n(n+1)} - \frac{N}{n(n+1)} - \frac{1}{N} \right] h
\end{aligned} \tag{D.2}$$

Therefore, we have

$$\begin{aligned}
f_i(x+h) - f_i(x) &= h \left(\frac{N-2n+1}{(n-1)n} + \frac{1}{N} \right) \\
&= h \left[\frac{N-n}{(n-1)n} - \frac{1}{n} + \frac{1}{N} \right] \\
&= h \frac{N-n}{n} \left(\frac{1}{n-1} - \frac{1}{N} \right) > 0,
\end{aligned} \tag{D.3}$$

and

$$\begin{aligned}
f_i(x) - f_i(x - h) &= \left[\frac{N}{n(n+1)} + \frac{1}{N} - \frac{1}{n+1} - \frac{1}{n} \right] h \\
&= \left[\frac{N-n}{n(n+1)} - \frac{N-n}{Nn} \right] h \\
&= \frac{N-n}{n} \left(\frac{1}{n+1} - \frac{1}{N} \right) h > 0.
\end{aligned} \tag{D.4}$$